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Sets, complements and boundaries

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ABSTRACT

The relations among a set, its complement, and its boundary are examined constructively. A crucial tool is a theorem that allows the construction of a point where a segment comes close to the boundary of a set in a Banach space. Brouwerian examples show that many of the results are the best possible.

1. INTRODUCTION

In this paper we investigate such questions as:

If we can compute the distance of any point from a subset of a metric space, can we compute the distance of any point from its boundary?

If a point is bounded away from the complement of a set, can we show that it is in the set?

The constructive answers to such questions contain significant information and reveal surprising connections that are hidden from a classical analysis.

We operate within the framework of Bishop's constructive analysis. The reader is assumed to be familiar with Chapters 1–4 and 6 of [2]. Additional background in constructive mathematics can be found in [1], [4], and [8]. For information about the recursive model of constructive mathematics see [6].

It is convenient to gather together some of the basic notions about subsets of a metric space (X, ρ). A set is *inhabited* if there exists an element in it – this terminology is preferred by some to *nonempty* in that it sounds more positive. An inhabited set S is *located* if

$$\rho(x, S) \equiv \inf\{\rho(x, s) : s \in S\}$$

exists for each $x \in X$. If S is located, then

$$-S = \{x \in X : \rho(x, S) > 0\}.$$

It is natural to consider the empty set located, with $\rho(x, \emptyset) \equiv \infty$ for each $x \in X$. Below we will extend this further to sets that aren't known to be empty or to be inhabited.

It is convenient to talk about $\rho(x, S)$ even if S is not located. Here is how we will use this expression.

$\rho(x, S) < r$ means that $\rho(x, s) < r$ for some s in S,

$\rho(x, S) \geq r$ means that $\rho(x, S) < r$ is impossible, that is, that $\rho(x, s) \geq r$ for all s in S,

$\rho(x, S) \leq \rho(x, S')$ means that $\rho(x, S) < r$ whenever $\rho(x, S') < r$,

$\rho(x, S) = \rho(x, S')$ means $\rho(x, S) \leq \rho(x, S')$ and $\rho(x, S') \leq \rho(x, S)$.

It is not hard to show that $\rho(x, S)$ exists (as a real number) for an inhabited set S if and only if for each $r' < r$, either $r' \leq \rho(x, S)$ or $\rho(x, S) < r$. (See the constructive least-upper-bound principle, [2, Chapter 2, (4.3)].) We take this latter condition to be the definition of 'located' for an arbitrary set S:

An arbitrary subset S of a metric space X is *located* if for each pair $r' < r$ of real numbers, and each $x \in X$, either $r' \leq \rho(x, S)$ or $\rho(x, S) < r$.

It is easy to see that S is located if and only if $\rho(x, S)$ can be thought of as an extended nonnegative real number – that is, an element of the one-point compactification of $[0, \infty)$ – for each x. To illustrate this definition, consider the located set S defined by taking a decreasing binary sequence (a_n) and setting

$$S \equiv \bigcap_n (na_n, \infty).$$

In the same way, we talk about

$$\rho(S, T) \equiv \inf\{\rho(s, t) : s \in S \text{ and } t \in T\}$$

for arbitrary subsets S and T; note that this is not the Hausdorff metric. We refer to objects like $\rho(x, S)$ and $\rho(S, T)$ as *distance expressions*. Each distance expression λ is determined by the set $\{r \in \boldsymbol{R} : \lambda < r\}$, and may be thought of as the (possibly empty) upper set of a Dedekind cut. Indeed the notion of a distance expression may be identified with that of an open upper set of positive real (or rational) numbers: if S is such a set, then $\rho(0, S) = S$. Distance expressions differ from the bounded extended reals of [8] in that they need not be strongly monotonic.

If x is a point and $r > 0$ is a real number, then $B(x, r)$ denotes the open ball of radius r centered at x. In general, if S in an arbitrary set, then

$$B(S, r) \equiv \{y : \rho(y, S) < r\}.$$

The *closure* of S is $\bar{S} \equiv \{x \in X : \rho(x, S) = 0\}$. The *interior* of S is

$$S^o \equiv \{x \in X : B(x, \varepsilon) \subset S \text{ for some } \varepsilon > 0\}.$$

The *complement* of S is

$$\sim S \equiv \{x \in X : \rho(x, s) > 0 \text{ for each } s \in S\};$$

the *metric complement* of S is

$$-S \equiv \{x \in X : \rho(x, S) > 0\} = (\sim S)^o.$$

We write $S \sim \Omega$ and $S - \Omega$ for $S \cap \sim\Omega$ and $S \cap -\Omega$ respectively. The *boundary* of S is

$$\partial S \equiv \bar{S} \cap \overline{\sim S}.$$

2. COHERENT SETS

We will look at three ways that a point might get into a set by virtue of distancing itself from the complement or the boundary, and clarify their nature with Brouwerian examples.[1]

Let S and Ω be subsets of a metric space (X, ρ). We say that S is *well contained in* Ω, and we write $S \subset\subset \Omega$, if $B(S, r) \subset \Omega$ for some $r > 0$. Note that if $S \subset\subset \Omega$, then $S \subset\subset \Omega^o$. If a subset of an open set Ω has the Heine–Borel property – every cover by open sets has a finite subcover – then it is well contained in Ω. In the recursive model of constructive mathematics, the Heine–Borel theorem is demonstrably false, and there is a compact subset of the open ball in $\boldsymbol{R}^2$ which has points arbitrarily close to the boundary of the ball – see [4], Chapter 6, (2.11).

We say that Ω is

weakly coherent if $x \in \Omega$ for each point x of $\bar{\Omega}$ that is bounded away from $\sim\Omega$;

edge coherent if $x \in \Omega$ for each point x of $\bar{\Omega}$ that is bounded away from $\partial\Omega$; and

coherent if $x \in \Omega$ whenever x is bounded away from $\sim\Omega$.

By definition, Ω is coherent if $-\sim\Omega \subset \Omega$. It follows that Ω is coherent if and only if $\Omega^o = -\sim\Omega$, and that an open set is coherent if and only if it is a metric complement.

If Ω is either coherent of edge coherent, then it is weakly coherent. If Ω is weakly coherent, then for each $r > 0$,

$$\{x \in \bar{\Omega} : \rho(x, \sim\Omega) \geq r\} \subset\subset \Omega;$$

if Ω is edge coherent, then

$$\{x \in \bar{\Omega} : \rho(x, \partial\Omega) \geq r\} \subset\subset \Omega.$$

[1] For the nature and role of Brouwerian examples in constructive mathematics, see Chapter 1 of [4].

If Ω is *located* and weakly coherent, then Ω is coherent, for if x is not in $\sim\Omega$, then $\rho(x, \Omega) > 0$ is impossible, so $\rho(x, \Omega) = 0$; whence $x \in \bar{\Omega}$ and, by weak coherence, $x \in \Omega$. We shall show in Proposition 14 that if Ω is a located weakly coherent subset of a *Banach space*, then Ω is edge coherent. So these three versions of coherence are equivalent for a located set in a Banach space.

Classically, every subset of a metric space is edge coherent and coherent. The following examples show why these coherence properties cannot be established constructively for arbitrary subsets. The first example is a very well-behaved set that is not even weakly coherent. The second is a set that is edge coherent but not coherent. The third is a coherent set that is not edge coherent. The fourth is a weakly coherent set that is neither edge coherent nor coherent.

Brouwerian Example 1. A totally bounded open subset Ω of $\boldsymbol{R}$ such that $\partial\Omega$ is finite and $\sim\Omega$ is located, but Ω is not weakly coherent.

Let P be any proposition, and define the open subset Ω of $\boldsymbol{R}$ by

$$\Omega \equiv (0,1) \cup (1,2) \cup \{x \in (0,2) : P \vee \neg P\}.$$

Clearly, Ω is dense in $(0,2)$ and is therefore totally bounded. Since $\neg(P \vee \neg P)$ is impossible, $\partial\Omega = \{0,2\}$ and

$$\sim\Omega = (-\infty, 0] \cup [2, \infty),$$

so 1, which belongs to $\bar{\Omega}$, is bounded away from $\sim\Omega$. But if $1 \in \Omega$, then $P \vee \neg P$.

Brouwerian Example 2. An inhabited, edge coherent, open subset of $\boldsymbol{R}$ that has finite boundary, located complement, and is not coherent.

Let P be any proposition, and define

$$\Omega \equiv (0,1) \cup (2,3) \cup \{x \in (1,2) : P \vee \neg P\}.$$

Then Ω is inhabited and open, $\sim\Omega = (-\infty, 0] \cup [3, \infty) \cup \{1, 2\}$ and $\partial\Omega = \{0, 1, 2, 3\}$. If $x \in \bar{\Omega}$ is bounded away from $\partial\Omega$, then x belongs to one of the intervals $(0,1), (1,2), (2,3)$ and therefore to Ω. (If $x \in \bar{\Omega} \cap (1,2)$, then $\Omega \cap (1,2)$ is inhabited, so $P \vee \neg P$, and therefore $x \in \Omega$.) Thus Ω is edge coherent. If, however, Ω were coherent, then $\frac{3}{2}$ would belong to Ω and we would have $P \vee \neg P$.

In our next Brouwerian example we will assume that every function from $\boldsymbol{N}$ to $\boldsymbol{N}$ is recursive; this is Church's thesis in a constructive setting. So we are working in the recursive constructive mathematics of the Russian School founded by Markov; see Chapter 3 of [4]. This enables us to construct (strong) *Specker sequences*: strictly monotone, bounded sequences of rational numbers that are eventually bounded away from any given real number.

Proposition 3 [recursive example]. *If $u < v$ are real numbers, then there is an inhabited open subset J of (u, v) such that ∂J and $\partial(-J)$ are empty.*

Proof. It suffices to construct such a set in $(-1,1)$. Let (r_m) be a positive increasing Specker sequence. By scaling we may assume that $r_m < 1$ for each m. Clearly $J \equiv \bigcup_{m=1}^{\infty}(-r_m, r_m)$ is an inhabited open subset of $(-1,1)$. If $x \in R$, then there exists M and $\delta > 0$ such that $||x| - r_m| \geq \delta$ for $m \geq M$. So $\bar{J} = J$, and $\sim J$ is open, hence equal to $-J$. That makes both J and $-J$ simultaneously open and closed, so each has empty boundary. □

Brouwerian Example 4 [uses Church's thesis]. An inhabited coherent bounded open subset of $\boldsymbol{R}$ that has finite boundary and is not edge coherent.

By Proposition 3 we can construct, for each positive integer n, an inhabited open subset I_n of $(1/(n+1), 1/n)$ with empty boundary. Let (a_n) be a binary sequence with at most one term equal to 1. Then

$$\Omega \equiv (-1,1) - \bigcup \{I_n : a_n = 1\},$$

being a metric complement, is coherent and open. It is also inhabited. If $a_n = 1$, then $\Omega = (-1,1) - I_n$ and $\partial\Omega = \{-1,1\}$; so if there exists $x \in \partial\Omega - \{-1,1\}$, then $a_n = 0$ for all n, $\Omega = (-1,1)$, and therefore $\partial\Omega = \{-1,1\}$, a contradiction. Hence $\partial\Omega = \{-1,1\}$.

Now suppose that Ω is edge coherent. Then 0 is in $\tilde{\Omega}$ because $(-1,0) \subset \Omega$, and 0 is bounded away from $\partial\Omega = \{-1,1\}$, so $0 \in \Omega$. Choose a positive integer N such that $(-1/N, 1/N) \subset \Omega$. If $a_n = 0$ for all $n \leq N$, then $a_n = 0$ for all n. So we could prove

$$\forall n(a_n = 0) \quad \text{or} \quad \exists n(a_n = 1).$$

Note that in this example, Ω is not located. Note also that $\bigcup \{I_n : a_n = 1\}$ is open and closed, but its complement is not open and its metric complement is not closed.

Brouwerian Example 5 [uses Church's thesis]. An inhabited weakly coherent bounded open subset of $\boldsymbol{R}$ that has finite boundary and is neither coherent nor edge coherent.

Again we assume Church's thesis. Take the set Ω from the previous example and translate it by -2, to obtain an inhabited coherent (and therefore weakly coherent) bounded open subset Ω_1 of $(-3,-1)$ that has finite boundary and is not edge coherent. Let Ω_2 be the set constructed in Brouwerian Example 2: an inhabited, edge coherent (and therefore weakly coherent), open subset of $(0,3)$ that has finite boundary and is not coherent. The set we want is $\Omega_1 \cup \Omega_2$.

3. CROSSING THE BOUNDARY OF A SET

One form of the *intermediate value theorem* states that if f is a uniformly continuous function on $[0,1]$, and $f(a) < 0 < f(b)$ for $0 < a < b < 1$, then there exists c in $[a,b]$ such that $f(c) = 0$. The intermediate value theorem is constructively equivalent to Bishop's omniscience principle LLPO (see Chapter 1 of [4]), and to the statement that for any two real numbers x and y, either $x \leq y$ or $y \leq x$. Indeed the usual interval-halving argument for the intermediate value theorem relies on this property of real numbers, which is recursively refutable in the sense that there exist recursive sequences (x_n) and (y_n) of recursive real numbers such that there is no binary recursive sequence (a_n) with the property that if $a_n = 0$, then $x_n \leq y_n$, and if $a_n = 1$, then $y_n \leq x_n$.

We can't find c such that $f(c) = 0$, but we can find c such that $f(c)$ is arbitrarily close to 0. That is, we can show that $\rho(0, f([a,b])) = 0$. The same considerations apply when traveling from a set to its complement along a straight line – we can't hope to find a point where we cross the boundary, but we might be able to find points where we get arbitrarily close to the boundary. In fact, finding intermediate values is a special case of finding boundary crossings on straight-line paths in $\boldsymbol{R}^2$.

Proposition 6. *Let* $f : [0,1] \to \boldsymbol{R}$ *be uniformly continuous,* $\sup f < B$, *and* $0 < a < b < 1$ *such that* $f(a) < 0 < f(b)$. *Then*

$$\Omega \equiv \{(x,y) \in \boldsymbol{R}^2 : 0 < x < 1 \quad \text{and} \quad f(x) < y < B\}$$

is totally bounded, $(a,0) \in \Omega$, *and* $(b,0) \in -\Omega$. *If* $(x,0)$ *is on the segment joining* $(a,0)$ *and* $(b,0)$, *and also in* $\partial\Omega$, *then* $f(x) = 0$.

Proof. If $0 < x' < 1$, then $(x', y) \in \Omega$ if and only if $f(x') < y$, and $(x', y) \in -\Omega$ if and only if $f(x') > y$. So if $(x,0) \in \partial\Omega$, then $f(x) = 0$. □

We are mostly interested in $\boldsymbol{R}^N$, but we can talk about straight-line paths in any linear space. For points x, y in a normed linear space we write

$$[x,y] \equiv \{tx + (1-t)y : 0 \leq t \leq 1\}.$$

If $x \neq y$, that is, if $\|x - y\| > 0$, then $[x,y]$ is closed.

The following lemma is the key tool for finding approximate boundary crossings. If λ and μ are distance expressions, then $\lambda \vee \mu$ denotes their supremum, which may be described by the equality $\{r \in \boldsymbol{R} : r > \lambda \vee \mu\} = \{r \in \boldsymbol{R} : r > \lambda \text{ and } r > \mu\}$.

Lemma 7. *Let* U *and* V *be subsets of a Banach space such that* $U \cup V$ *is dense.*

(i) *If* $u_0 \in U$ *and* $v_0 \in V$, *then* $\rho([u_0, v_0], \bar{U} \cap \bar{V}) = 0$.

(ii) $\rho(x, \bar{U} \cap \bar{V}) = \rho(x, U) \vee \rho(x, V)$.

Proof. We want to show that $\rho([u_0, v_0], \bar{U} \cap \bar{V})$ is small, so we construct an element of $\bar{U} \cap \bar{V}$ that is close to $[u_0, v_0]$ by approximate interval halving. Choose $0 < \varepsilon < \frac{1}{2}$. Given u_{n-1} and v_{n-1}, we construct u_n and v_n such that

1. $[u_n, v_n] \subset B([u_{n-1}, v_{n-1}], \varepsilon^n)$,
2. $|u_n - v_n| < \frac{1}{2}|u_{n-1} - v_{n-1}| + \varepsilon^n$,
3. $|u_n - u_{n-1}| + |v_n - v_{n-1}| < \frac{1}{2}|u_{n-1} - v_{n-1}| + \varepsilon^n$.

To do this choose w in $U \cup V$ within ε^n of $\frac{1}{2}(u_{n-1} + v_{n-1})$. If $w \in U$, set $u_n = w$ and $v_n = v_{n-1}$; if $w \in V$, set $u_n = u_{n-1}$ and $v_n = w$. Let $c \equiv |u_0 - v_0| + 2\varepsilon/(1-2\varepsilon)$. From (2) we get

$$|u_n - v_n| < \frac{c}{2^n};$$

so from (3) we get

$$|u_n - u_{n-1}| + |v_n - v_{n-1}| < \frac{c}{2^n} + \varepsilon^{n-1},$$

which shows that u and v are Cauchy sequences. From (2) they have a common limit in $\bar{U} \cap \bar{V}$ and, from (1), this limit is within $\varepsilon/(1-\varepsilon)$ of $[u_0, v_0]$. This proves (i). Conclusion (ii) follows because if z is in $[u_0, v_0]$, then $|x - z| \leq |x - u_0| \vee |x - v_0|$. □

We shall say that a subset Ω of a normed linear space has the *boundary crossing property* if $\rho(\partial\Omega, [x_0, y_0]) = 0$ whenever $x_0 \in \Omega$ and $y_0 \in \sim\Omega$.

Proposition 8. *Let Ω be a subset of a Banach space such that $\Omega \cup \sim\Omega$ is dense. Then Ω has the boundary crossing property.*

Proof. Apply Lemma 7 with $U = \Omega$ and $V = \sim\Omega$. □

Corollary. *Any located subset of a Banach space has the boundary crossing property.*

Proof. If Ω is a located subset of a Banach space, then $\Omega \cup -\Omega$ is dense. So $\Omega \cup \sim\Omega$ is dense, and we can apply Proposition 8. □

What about the boundary crossing property for straight line paths in complete metric spaces other than Banach spaces?

Brouwerian Example 9. There exists an open, totally bounded subset Ω of a compact subspace of $\boldsymbol{R}^2$ that does not have the boundary crossing property for straight line paths.

Consider the compact subspace X of $\boldsymbol{R}^2$ defined to be the closure of

$$\{(x, 0) : 0 \leq x \leq 1\} \cup \left\{\left(\frac{k}{n}, \frac{1}{n}\right) : k = 0, 1, \ldots, n;\ n = 1, 2, \ldots\right\}.$$

Let a_n be an increasing binary sequence, and define Ω to be the union of the three open sets

$$\left\{(x, y) \in X : x < \frac{1}{3}\right\}$$

$$\left\{\left(\frac{k}{n}, \frac{1}{n}\right) \in X : \frac{k}{n} < \frac{2}{3} \text{ and } a_n = 0\right\}$$

$$\left\{(x,0) \in X : x < \frac{2}{3} \text{ and } a_n = 0 \text{ for all } n\right\}.$$

Then Ω is open and totally bounded, and $\partial\Omega \subset \{(x,0) : 0 \leq x \leq 1\}$. Suppose that $(x,0) \in \partial\Omega$. If $x < \frac{2}{3}$, then $a_n = 0$ for all n is impossible, while if $x > \frac{1}{3}$, then $a_n = 0$ for all n. Thus we can't find an element of $\partial\Omega$. But $(0,0) \in \Omega$ and $(1,0) \in {\sim}\Omega$ are joined by a straight line in X.

There are a number of applications of the boundary crossing property. First a lemma about distance expressions.

Lemma 10. *Let λ and μ be distance expressions. If $\lambda \wedge \mu$ and $\lambda \vee \mu$ are extended real numbers, then so are λ and μ.*

Proof. Let s and t be extended real numbers such that $s = \lambda \wedge \mu$ and $t = \lambda \vee \mu$. To show that λ is an extended real, let r, r' be real numbers with $r < r'$. If $t < r'$, then $\lambda < r'$, so we may assume that $t > r$. If $s > r$, then $\lambda > r$, so we may assume that $s < t$. Because $\lambda \wedge \mu = s < t$, either $\lambda < t$ or $\mu < t$. In the former case, $\mu = t$ and $\lambda = s$, in the latter, $\lambda = t$ and $\mu = s$. □

Proposition 11. *Let Ω be a subset of a Banach space. Then Ω and ${\sim}\Omega$ are located if and only if $\partial\Omega$ is located and $\Omega \cup {\sim}\Omega$ is dense.*

Proof. Let $A \equiv \Omega$ and $B \equiv {\sim}\Omega$. As $A \cup B$ is dense in either case, Lemma 7 applies and

$$\rho(x, \bar{A} \cap \bar{B}) = \rho(x, A) \vee \rho(x, B)$$

for all x. In view of Lemma 10, this condition implies that A and B are located if and only if $A \cup B$ and $\bar{A} \cap \bar{B}$ are located. Note that if $B = {\sim}A$, then $A \cup B$ is dense if and only if it is located. □

Proposition 12. *Let Ω be a coherent subset of a Banach space. If both $\partial\Omega$ and ${\sim}\Omega$ are located, then Ω is located.*

Proof. This time, ${\sim}\Omega \cup -({\sim}\Omega)$ is dense. Coherence of Ω means that $-({\sim}\Omega) \subset \Omega$, so $\Omega \cup {\sim}\Omega$ is dense. The desired conclusion follows from this and Proposition 11. □

Brouwerian Example 2 shows that we cannot drop the coherence hypothesis from Proposition 12. The next example shows that we cannot drop the hypothesis that ${\sim}\Omega$ is located.

Brouwerian Example 13. A bounded, edge coherent, coherent open subset Ω of $\boldsymbol{R}$ such that $\partial\Omega$ is located but Ω is not located.

Let P be any proposition, and define

$$\Omega \equiv \left\{ x : x \in \left(\frac{1}{2n}, \frac{1}{2n-1} \right), \text{ or } x \in \left(\frac{1}{2n+1}, \frac{1}{2n} \right) \text{ and } \neg P, \text{ for some } n \right\}.$$

Then $\partial\Omega$ is the closure of $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and

$$-\partial\Omega = \cdots \cup \left(\frac{1}{4}, \frac{1}{3} \right) \cup \left(\frac{1}{3}, \frac{1}{2} \right) \cup \left(\frac{1}{2}, 1 \right) \cup (-\infty, 0) \cup (1, \infty).$$

Clearly Ω is edge coherent. To see that Ω is coherent, note that if $x \in (1/(2n+1), 1/(2n))$ is bounded away from $\sim\Omega$, then $\neg P$, so $x \in \Omega$. If $\rho(\frac{5}{12}, \Omega) < \frac{1}{12}$, then $\neg P$, while if $\rho(\frac{5}{12}, \Omega) > 0$, then $\neg\neg P$, so if Ω were located, we could prove the *weak law of excluded middle*:

$$\neg\neg P \vee \neg P,$$

which is equivalent to *DeMorgan's Law*: $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$. By Proposition 12, we could also prove this if $\sim\Omega$ were located.

Proposition 14. *If Ω is a weakly coherent subset that has the boundary crossing property, then Ω is edge coherent.*

Proof. Let x be a point of $\bar{\Omega}$ and r a positive number such that $\rho(x, \partial\Omega) \geq r$. It follows from the boundary crossing property that $\rho(x, \sim\Omega) \geq r$; whence $x \in \Omega$ by the weak coherence of Ω. □

Proposition 15. *If Ω is a Lebesgue measurable subset of $\boldsymbol{R}^N$ with located boundary, then Ω is located.*

Proof. Since Ω is Lebesgue measurable, $\Omega \cup \sim\Omega$ is a full set. By [2] (Chapter 6, (3.4)), every ball of positive radius intersects $\Omega \cup \sim\Omega$, which is therefore dense. The desired conclusion follows from this and Proposition 11. □

Brouwerian Example 16. A Lebesgue integrable, edge coherent, located open subset of $\boldsymbol{R}^2$ whose boundary is not located.

Let P be any proposition, and define

$$\Omega \equiv (B(0,1) - \{0\}) \cup \{x \in B(0,1) : \neg P\}.$$

Then Ω is open, located, and Lebesgue integrable. It is also edge coherent: for if $x \in \bar{\Omega}$ and $\|x - y\| \geq r > 0$ for all $y \in \partial\Omega$, then either $0 < \|x\| < 1$ or $\|x\| < r$; in the latter case, $0 \notin \partial\Omega$, so P is impossible, $\Omega = B(0, 1)$, and therefore $x \in \Omega$. But if $\partial\Omega$ were located, then, by considering $\rho(0, \partial\Omega)$, we could prove $\neg\neg P \vee \neg P$.

It is a trivial classical result that if x belongs to a subset Ω of a normed linear space, and if y is a closest point to x in $\partial\Omega$, then $tx + (1-t)y \in \Omega$ for

$0 < t \leq 1$. Constructively, we have to put some additional hypothesis on Ω, as the Brouwerian example

$$\Omega \equiv \{-1, 0, 1\} \cup \{x : -1 < x < 1 \text{ and } P \vee \neg P\}$$

shows. The following proposition also gives information when $y \in \partial\Omega$ is close to x, but not necessarily the closest point in $\partial\Omega$ to x (which we may not be able to find).

Proposition 17. *Let Ω be a coherent subset of a Banach space such that $\Omega \cup {\sim}\Omega$ is dense and $\partial\Omega$ is located. If $x \in \bar{\Omega}$ and $d \equiv \rho(x, \partial\Omega) > 0$, then $B(x, d) \subset \Omega$. It follows that if $y \in \partial\Omega$ and $\|x - y\| < d + \varepsilon$, then $x_t \equiv tx + (1 - t)y \in \Omega$ whenever $\varepsilon/(d + \varepsilon) \leq t \leq 1$.*

Proof. Lemma 7 gives $\rho(x, \partial\Omega) = \rho(x, {\sim}\Omega)$. If $\|x - x'\| < d$, then x' is bounded away from ${\sim}\Omega$, so coherence gives $x' \in \Omega$.
For the second part, compute

$$\|x_t - x\| = (1 - t)\,\|x - y\| \leq (1 - t)(d + \varepsilon) \leq d.$$

The first inequality is strict if $t < 1$, and the second is strict if $t > \varepsilon/(d + \varepsilon)$. □

One consequence of Proposition 17 is a result with applications in the constructive theory of elliptic partial differential equations (see [9]).

Proposition 18. *Let Ω be a coherent bounded subset of $\boldsymbol{R}^N$ such that $\Omega \cup {\sim}\Omega$ is dense and $\partial\Omega$ is located. Let $u : \bar{\Omega} \to \boldsymbol{R}$ be uniformly continuous on $\bar{\Omega}$, differentiable on Ω, and vanish on $\partial\Omega$. If $\|\nabla u(x)\| \leq M$ for all $x \in \Omega$, then*

$$|u(x)| \leq M\rho(x, \partial\Omega) \quad (x \in \bar{\Omega}).$$

Proof. Let $d \equiv \rho(x, \partial\Omega)$ and suppose $|u(x)| > Md$. Then $\rho(x, \partial\Omega) > 0$ because u is uniformly continuous and vanishes on $\partial\Omega$. Proposition 17 shows that $B(x, d) \subset \Omega$. Suppose y and z are in $B(x, d)$. We may assume without loss of generality that $z - y$ is parallel to the N^{th} coordinate axis. So

$$\begin{aligned}
|u(z) - u(y)| &= \left| \int_{y_N}^{z_N} \frac{\partial}{\partial \xi} u(y_1, \ldots, y_{N-1}, \xi)\, \mathrm{d}\xi \right| \\
&\leq \int_{y_N}^{z_N} \left| \frac{\partial}{\partial \xi} u(y_1, \ldots, y_{N-1}, \xi) \right| \mathrm{d}\xi \\
&\leq M|z_N - y_N| \\
&\leq M\|z - y\|.
\end{aligned}$$

Thus the function u is M-Lipschitz on $B(x, d) \subset \Omega$, hence on $\bar{B}(x, d) \subset \bar{\Omega}$. As $u(y) = 0$ for $y \in \partial\Omega \subset \bar{\Omega}$, it follows that $|u(x)| \leq M\|x - y\|$ for each $y \in \partial\Omega$, so $|u(x)| \leq Md$. □

Let Ω be a subset of a metric space (X, ρ). We say that K *approximates* Ω *internally to within* ε if $K \subset\subset \Omega$, and $\rho(x, \partial\Omega) < \varepsilon$ for each x in $\Omega - K$. If, for each $\varepsilon > 0$, the set Ω can be approximated internally to within ε by a set of type T, then we say that Ω is *approximated internally by sets of type T.* Given such a set Ω, we denote by K^{ε} a set of type T that approximates it to within ε. If the closure of a set of type T is again of type T, then we may assume that K^{ε} is closed. We will be interested in sets that can be approximated internally by located sets.

Note that if $\Omega \subset \partial\Omega$, then Ω is approximated internally by compact sets (take K^{ε} empty). This observation gives rise to examples where neither Ω nor $\sim\Omega$ can be shown to be inhabited: if P is any proposition, and $\boldsymbol{Q}$ is the set of rational numbers, consider

$$\Omega \equiv \{x \in \boldsymbol{Q} : P\} \cup \{x \in \sim\boldsymbol{Q} : \neg P\}.$$

Here is another class of examples.

Proposition 19. *Let Ω be a subset of $\boldsymbol{R}^N$ such that $\Omega^o = -L$ for some located set $L \subset \sim\Omega$. Then Ω is approximated internally by located sets.*

Proof. As $\Omega^o \cup L$ is dense, and $L \subset \sim\Omega$, we have $\rho(x, \partial\Omega) \leq \rho(x, \Omega^o) \vee \rho(x, L)$. The set $K(t) \equiv \{x \in \boldsymbol{R}^N : \rho(x, L) \geq t\}$ is located for all but countably many $t > 0$. If $t < \varepsilon$, and $x \in \Omega - K(t)$, then $\rho(x, L) < \varepsilon$, so $\rho(x, \partial\Omega) < \varepsilon$. □

Note that if Ω is open, then the hypothesis that $L \subset \sim\Omega$ is superfluous, so a metric complement of a located set in $\boldsymbol{R}^N$ is approximated internally by located sets. In particular, an open set that is approximated internally by compact sets need not be located. Let (a_n) be an increasing binary sequence, and

$$L = \{x \in \boldsymbol{R} : |x| \geq a_n/n \text{ for all } n\}.$$

Then $\Omega \equiv -L$ is $(-(1/n), 1/n)$ if $a_n = 1 + a_{n-1}$.

An open set can be approximated internally by compact sets without being bounded – for example, $\bigcup(n - 1/n, n + 1/n)$.

Proposition 19 has a converse, at least for open sets.

Proposition 20. *Let Ω be a subset of $\boldsymbol{R}^N$ that is approximated internally by located sets. Then $\sim\Omega^o$ is located and $\Omega^o = -\sim\Omega^o$.*

Proof. We first show, for $r > 0$, that if $\rho(x, K^r) > 0$, then $\rho(x, \sim\Omega^o) < r$. As $K^{r/n} \subset\subset \Omega$ for each positive integer n, we can construct a binary sequence (λ_n) so that $\lambda_1 = 0$,

if $\lambda_n = 0$ then $\rho(x, K^{r/n}) > 0$, and

if $\lambda_n = 1$, then $x \in \Omega$.

If $\lambda_n = \lambda_{n-1} + 1$, then $x \in \Omega \sim K^{r/(n-1)}$, so $\|x - y_n\| < r/(n-1)$ for some $y_n \in \sim\Omega$. Construct a sequence (x_n) in $\boldsymbol{R}^N$ such that

$x_n = x$ if $\lambda_m = 0$ for all $m \le n$, and
$x_n = y_m$ if $m \le n$ is the first number such that $\lambda_m = 1$.

Clearly (x_n) is a Cauchy sequence that is well contained in $B(x,r)$. Let x_∞ be its limit in $B(x,r)$. If x_∞ were in Ω^o, then λ_n would have to be 0 for all n, so $\rho(x,K^{r/n}) > 0$ for all n, and so $x_\infty \notin \Omega^o$. Hence $x_\infty \notin \Omega^o$.

We will show that $\sim\Omega^o$ is located. Given x, a large real number $R > 0$, and a small real number $\varepsilon > 0$, choose $\delta > 0$ so that $B(K^\varepsilon, 2\delta) \subset \Omega$, and construct a finite δ-approximation A to $B(x,R)$. Partition A into disjoint subsets A_0 and A_1 such that $\rho(a,K^\varepsilon) < \delta$ if $a \in A_0$, and $\rho(a,K^\varepsilon) > 0$ if $a \in A_1$. Then $A_0 \subset \Omega^o$ and $A_1 \subset B(\sim\Omega^o,\varepsilon)$. Now let $d = \rho(x,A_1)$, which is either a real number or (if A_1 is empty) ∞. As $A_1 \subset B(\sim\Omega^o,\varepsilon)$, we have $\rho(x,\sim\Omega^o) < d+\varepsilon$. Suppose $y \in \sim\Omega^o \cap B(x,R)$, so $\rho(y,K^\varepsilon) \ge 2\delta$. Then there exists $a \in A$ within δ of y. If $a \in A_0$, then $\rho(y,K^\varepsilon) < 2\delta$, so $a \in A_1$. Hence $R \wedge (d-\delta) \le \rho(x,\sim\Omega^o)$. As we may assume that $\delta \le \varepsilon$, we have

$$R \wedge (d-\varepsilon) \le \rho(x,\sim\Omega^o) < d+\varepsilon.$$

To show that $\sim\Omega^o$ is located, let $r < r'$, $R \equiv \sup\{1,r'\}$, and $\varepsilon \equiv \frac{1}{3}(r'-r)$. If $d < r'-\varepsilon$, then $\rho(x,\sim\Omega^o) < r'$; if $d > r+\varepsilon$, then $r < \rho(x,\sim\Omega^o)$.

It remains to show that Ω^o is the metric complement of $\sim\Omega^o$. Suppose $\rho(x,\sim\Omega^o) = r > 0$. If $\rho(x,K^r) > 0$, then x cannot be in Ω, so $x \in \sim\Omega^o$, a contradiction. Thus $x \in K^r \subset \Omega^o$. □

Proposition 21. *If Ω is a subset of $\mathbf{R}^N$ that is approximated internally by located sets, then Ω has the boundary crossing property.*

Proof. Let x be in Ω, and y in $\sim\Omega$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $B(K^\varepsilon,2\delta) \subset \Omega$. If $\rho(x,K^\varepsilon) > 0$, then $\rho(x,\partial\Omega) < \varepsilon$. If $\rho(x,K^\varepsilon) < \delta$, set $x_t = (1-t)x+ty$ and $f(t) = \rho(x_t,K^\varepsilon)$. Then $f(1) \ge 2\delta$ and $f(0) < \delta$. Choose t so that $f(t) > 0$ and $f(t)$ is near δ. Then $x_t \in \Omega$ because $B(K^\varepsilon,2\delta) \subset \Omega$, and $x_t \in -K^\varepsilon$ because $f(t) > 0$. So $\rho(x_t,\partial\Omega) < \varepsilon$. □

Proposition 22. *Let Ω a subset of a metric space, and suppose that Ω is approximated internally by compact sets. Then Ω is totally bounded if and only if $\partial\Omega$ is totally bounded.*

Proof. Let $\varepsilon > 0$, and choose $r \in (0,\varepsilon)$ such that $B(K^\varepsilon,4r) \subset \Omega$. If Ω is totally bounded, let $\{x_1,\ldots,x_n\}$ be an r-approximation to Ω, and partition $\{1,\ldots,m\}$ into subsets I and J such that $\rho(x_i,K^\varepsilon) < r$ if $i \in I$, and $\rho(x_i,K^\varepsilon) > 0$ if $i \in J$. For each $i \in J$ there is $y_i \in \partial\Omega$ such that $\rho(x_i,y_i) < \varepsilon$. We will show that $\{y_i : i \in J\}$ is a 3ε-approximation to $\partial\Omega$.

Given $y \in \partial\Omega$, choose $x \in \Omega$ such that $\rho(y,x) < r$, and then choose i such that $\rho(x,x_i) < r$. If $i \in I$, then

$$\rho(y,K^\varepsilon) \le \rho(y,x)+\rho(x,x_i)+\rho(x_i,K^\varepsilon) < 3r$$

so $B(y,r) \subset B(K^{\varepsilon},4r) \subset \Omega$. Thus $y \in \Omega^{o}$, which is absurd. Hence $i \in J$. Moreover,

$$\rho(y, y_i) \leq \rho(y, x) + \rho(x, x_i) + \rho(x_i, y_i) < r + r + \varepsilon < 3\varepsilon,$$

so $\{ y_i : i \in J\}$ is a 3ε-approximation to $\partial\Omega$.

Now suppose that $\partial\Omega$ is totally bounded. Let A be a finite ε-approximation to $\partial\Omega$, and B be a finite ε-approximation to K^{ε}. Since for each $x \in \Omega$ either $\rho(x, K^{\varepsilon}) > 0$ or $\rho(x, K^{\varepsilon}) < \varepsilon$, it is clear that $A \cup B$ is a 2ε-approximation to Ω. □

Proposition 23. *If a subset of a metric space is approximated internally by located sets, then it is edge coherent.*

Proof. Let Ω be approximated internally by located sets. Let $x \in \bar{\Omega}$ be such that $\rho(x, \partial\Omega) \geq 2r > 0$. If $\rho(x, K^r) > 0$, then there exists $x' \in \Omega$ so close to x that $\rho(x', K^r) > 0$ and $\rho(x', \partial\Omega) \geq r$. This contradicts the properties of K^r, so $\rho(x, K^r) = 0$ and therefore $x \in K^r \subset \Omega$. □

Proposition 24. *If Ω is an edge-coherent totally bounded subset of a metric space, and Ω has totally bounded boundary, then Ω is approximated internally by compact sets.*

Proof. Given $\varepsilon > 0$, choose $r \in (0, \varepsilon)$ such that

$$K \equiv \{x \in \bar{\Omega} : \rho(x, \partial\Omega) \geq r\}$$

is compact ([2], Chapter 4, (4.9)). Since Ω is edge coherent, $B(K, r/2) \subset \Omega$, so $K \subset\subset \Omega$. On the other hand, if $x \in \Omega - K$, then $\rho(x, \partial\Omega) \leq r < \varepsilon$. □

5. THE EXTERIOR POI CONDITION

When Ω is compact, $\partial\Omega \subset \overline{-\Omega}$ because $\sim\Omega = -\Omega$. The condition $\partial\Omega \subset \overline{-\Omega}$ is interesting in its own right. It is straightforward to deduce from it that $\partial\Omega = \partial\bar{\Omega}$. It is also easy to see that $\partial\Omega \subset \overline{-\Omega}$ is equivalent to the following condition:

If $x \in \partial\Omega$ and $\varepsilon > 0$, then for some $\delta > 0$ there is a δ-ball contained in $\sim\Omega$ that is within ε of x.

If we think of the δ-ball as attached to x by a string of length less than ε, then we have pictured a *poi* (the Maori term for such an object). If, given $\varepsilon > 0$, we can choose δ independent of x, then we say that Ω satisfies the *exterior poi condition.*

We get the uniformity required by the exterior poi condition if the boundary is totally bounded.

Proposition 25. *If $\partial\Omega$ is totally bounded and contained in $\overline{-\Omega}$, then Ω satisfies the exterior poi condition.*

Proof. If $x \in \partial\Omega$, then $\rho(x, y) < \varepsilon$ for some y in $-\Omega$. Let

$$\delta \equiv \inf\{\rho(x, \Omega), \varepsilon - \rho(x, y)\}.$$

Then $B(y, \delta) \subset \sim\Omega \cap B(x, \varepsilon)$.

Now let A be a finite ε-approximation to $\partial\Omega$, and choose $\delta > 0$ so that $\sim\Omega \cap B(a, \varepsilon)$ contains a δ-ball for each a in A. Then $\sim\Omega \cap B(x, 2\varepsilon)$ contains a δ-ball for each x in $\partial\Omega$. □

In particular, if both Ω and $\partial\Omega$ are compact, then Ω satisfies the exterior poi condition. (Classically, it suffices to assume Ω is compact.) It is not sufficient, even classically, to assume only that Ω is totally bounded with compact boundary – for example, $\Omega \equiv (-1, 0) \cup (0, 1)$.

We can get a partial converse to Proposition 25.

Proposition 26. *Let Ω be a totally bounded subset of $\boldsymbol{R}^N$ that satisfies the exterior poi condition. Then $\partial\Omega$ is compact.*

Proof. For any $\varepsilon > 0$, let $\delta > 0$ be given by the exterior poi condition. We will approximate $\partial\Omega$ within 2ε. Choose a finite approximation A to a big ball containing Ω so that A intersects any open ball of radius $\delta/2$ contained in the big ball. Partition A into disjoint subsets A_0 and A_1 so that if $a \in A_0$, then $0 < \rho(a, \Omega) < \varepsilon$ while if $a \in A_1$, then $\rho(a, \Omega) < \delta/2$ or $\rho(a, \Omega) > \varepsilon - \delta/2$. By boundary crossing, for each $a \in A_0$ there is $b_a \in \partial\Omega$ such that $\|b_a - a\| < \varepsilon$. We shall show that each $x \in \partial\Omega$ is within 2ε of some b_a.

Choose y so that $B(y, \delta) \subset (\sim\Omega) \cap B(x, \varepsilon)$. Then there is a in $A \cap B(y, \delta/2)$. So $\|a - x\| < \varepsilon$ and $\delta/2 \leq \rho(a, \Omega) \leq \varepsilon - \delta/2$. Thus $a \in A_0$ and $\|b_a - x\| < 2\varepsilon$. □

The uniformity of the exterior poi condition, rather than just the condition that $\partial\Omega \subset \overline{-\Omega}$, is necessary for Proposition 26. Let (r_n) be an enumeration of the rational numbers in $(0, 1)$, starting with $r_1 = \frac{1}{2}$, and let (a_n) be a decreasing binary sequence. Let Ω be the closure of $\{a_n r_n : n = 1, 2, \ldots\}$. Then Ω is compact, so $\overline{-\Omega} = \overline{\sim\Omega} \supset \partial\Omega$. But if $\partial\Omega$ were located, then either $\rho(\frac{1}{2}, \partial\Omega) < \frac{1}{2}$, in which case there would be a point in $-\Omega$ in $(0, 1)$, and we could find n such that $a_n = 0$, or $\rho(\frac{1}{2}, \partial\Omega) > 0$, in which case $a_n = 1$ for all n.

We cannot interchange the rôles of Ω and $\partial\Omega$ in Proposition 26.

Proposition 27 [recursive counterexample]. *There is a coherent, edge coherent bounded open subset Ω of $\boldsymbol{R}^N$ that has empty boundary, and is not located.*

Proof. Let (r_n) be a strictly decreasing Specker sequence in $(0, 1)$, and define

$$A \equiv \{x \in \boldsymbol{R}^N : \|x\| > r_n \text{ for some } n\},$$

$$\Omega \equiv -A = \{x \in \boldsymbol{R}^N : \|x\| < r_n \text{ for all } n\}.$$

Then Ω is bounded. It is open and coherent because it is a metric complement. It is edge coherent because it is closed, and the boundary is empty. If it were located, then it would be totally bounded, so

$$\lim_{n\to\infty} r_n = \inf_{n\geq 1} r_n$$

would exist, which is impossible as every real number is eventually bounded away from (r_n). $\square$

A subset Ω of $\boldsymbol{R}^N$ satisfies the *exterior cone condition* if there exist $r, \theta > 0$ such that for each $x \in \partial\Omega$ there is a right circular cone C with vertex x, vertex angle θ, and height r such that $C \cap \bar{\Omega} = \{x\}$. The exterior cone condition rules out cusps on $\partial\Omega$ pointing into Ω, and plays an important rôle in solving the Dirichlet Problem on totally bounded open subsets of $\boldsymbol{R}^N$. The usual formulation of the exterior cone condition allows r and θ to vary with the point $x \in \partial\Omega$. However, a simple sequential compactness argument shows, classically, that r and θ can be chosen independent of x when $\partial\Omega$ is compact. Clearly the exterior cone condition implies the exterior poi condition.

It is a classical theorem that if a bounded open subset Ω of $\boldsymbol{R}^N$ satisfies the exterior cone condition, then for each uniformly continuous $f : \partial\Omega \to \boldsymbol{R}$ the Dirichlet Problem

$$(1) \qquad \begin{cases} \Delta u = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a continuous solution $u : \bar{\Omega} \to \boldsymbol{R}$ that is uniformly twice differentiable on each compact set K well contained in Ω [5]; in that case we say that the Dirichlet Problem has a *strong solution*. This suggests that the exterior cone condition, plus solvability of the Dirichlet Problem, may be connected with locating a bounded open set Ω in $\boldsymbol{R}^N$.

The coherent, nonlocated set Ω in Proposition 27 has empty boundary, so there is only one uniformly continuous function – the empty function – on that boundary, and the Dirichlet Problem (1) has infinitely many strong solutions.

The following example shows that adding the requirement that (1) have a *unique* solution, but dropping coherence, does not locate Ω.

Brouwerian Example 28. An inhabited, edge coherent, bounded, nonlocated open subset Ω of $\boldsymbol{R}^N$ that satisfies the exterior cone condition, such that $\partial\Omega$ is compact and the Dirichlet Problem (1) has a unique strong solution for each uniformly continuous $f : \partial\Omega \to \boldsymbol{R}$.

Let P be any proposition such that $\neg\neg P$, and let (r_n) be an increasing Specker sequence in $(\frac{1}{2}, 1)$. Define open subsets of $\boldsymbol{R}^N$ by

$$A \equiv \{x \in \boldsymbol{R}^N : r_n < \|x\| < 1 \text{ for all } n\},$$

$$B \equiv \{x \in \boldsymbol{R}^N : P \text{ and } \|x\| < 1\},$$

$$\Omega \equiv A \cup B.$$

It is easy to show that

$$\sim\Omega = \{x \in \boldsymbol{R}^N : \|x\| \geq 1\}$$

and

$$\partial\Omega = \{x \in \boldsymbol{R}^N : \|x\| = 1\}.$$

So $\partial\Omega$ is located and Ω satisfies the exterior cone condition. Given a point x of $\bar{\Omega}$ that is bounded away from $\partial\Omega$, and noting that $\|x\| < 1$, choose $\delta > 0$ and ν such that $|\|x\| - r_n| \geq \delta$ for all $n \geq \nu$. Then choose $\xi \in \Omega$ such that $\|x - \xi\| < \delta$. If $\xi \in A$, then

$$\|x\| \geq \|\xi\| - \|x - \xi\| > r_n - \delta$$

for all n, so $\|x\| \geq r_n + \delta$ for all $n \geq \nu$ and therefore $x \in A \subset \Omega$; if $\xi \in B$, then $x \in \Omega = B(0, 1)$. Hence Ω is edge coherent.

For a given uniformly continuous function $f : \partial\Omega \to \boldsymbol{R}$, the restriction to Ω of the solution of the Dirichlet Problem

$$\begin{aligned} &\Delta u = 0 \quad \text{on } B(0, 1), \\ &u(x) = f(x) \quad \text{if } \|x\| = 1 \end{aligned}$$

certainly solves the Dirichlet Problem (1) on $\bar{\Omega}$. This solution is given explicitly by the formula

$$u(x) = \frac{1 - \|x\|^2}{N\omega_N} \int_{\|\xi\|=1} \frac{f(\xi)}{\|x - \xi\|^N} \, \mathrm{d}S,$$

where $\mathrm{d}S$ denotes the element of surface on the boundary of the unit ball, and ω_N is the hypervolume of that ball ([5], Theorem 2.6). Now suppose that the Dirichlet Problem

$$(2) \qquad \begin{cases} \Delta u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution u that is nonzero at some point of Ω. If P holds, then $\Omega = B(0, 1)$ and (2) has the unique solution 0, a contradiction; so $\neg P$ holds, which is absurd. Hence (2) has the unique solution 0, and therefore (1) has a unique solution.

However, if Ω were located, then either $\rho(0, \Omega) > 0$ or $\rho(x, \Omega) < r_1$. The former is ruled out, as $-\Omega = -B(0, 1)$. Hence $\rho(x, \Omega) < r_1$, so B is inhabited and therefore P holds.

6. TWO RECURSIVE EXAMPLES

We can sometimes modify Brouwerian examples so that they become specific recursive examples that do not depend on an underlying proposition P or binary sequence (a_n). We will do this for Brouwerian Example 4. First we need to establish a couple of generalities about boundaries of metric complements.

Proposition 29. *If A is a subset of a located set L, then $\overline{\sim(-A)} \subset \bar{L}$ and therefore $\partial(-A) \subset \bar{L}$.*

Proof. Suppose, by way of contradiction, that $x \in \overline{\sim(-A)}$ and $\rho(x, L) > 0$. Then there exists $y \in \sim(-A)$ such that $\rho(y, L) > 0$ and therefore $y \in -A$, which is impossible. $\square$

Proposition 30. *Suppose L_1 and L_2 are located sets such that $\rho(L_1, L_2) > 0$. If $A_1 \subset L_1$ and $A_2 \subset L_2$, then*

$$\partial(-(A_1 \cup A_2)) = \partial(-A_1) \cup \partial(-A_2).$$

Proof. Suppose that $x \in \partial(-A_1) \subset \overline{L_1}$. Then $\rho(x, L_2) > 0$, so $x \in -A_2 \cap \overline{-A_1}$. Since $-A_2$ is open, it follows that $x \in \overline{-A_2 \cap -A_1}$; whence $x \in \overline{-(A_1 \cup A_2)}$. Also,

$$x \in \overline{\sim(-A_1)} \subset \overline{\sim(-(A_1 \cup A_2))},$$

so $x \in \partial(-(A_1 \cup A_2))$.

Conversely, suppose that $x \in \partial(-(A_1 \cup A_2))$; then $x \in \overline{-(A_1 \cup A_2)} \subset \overline{-A_1}$ and

$$x \in \overline{\sim(-(A_1 \cup A_2))} = \overline{\sim((-A_1) \cap (-A_2))}.$$

By Proposition 29,

$$x \in \overline{L_1 \cup L_2} = \overline{L_1} \cup \overline{L_2};$$

without loss of generality, we may assume that $x \in \overline{L_1}$. Then $x \in -A_2$, so $B(x, \varepsilon) \subset -A_2$ for some $\varepsilon > 0$. Choose y in $B(x, \varepsilon) \cap \sim((-A_1) \cap (-A_2))$; then $y \in -A_2 \cap \sim((-A_1) \cap (-A_2))$, so $y \in \sim(-A_1)$. Since ε can be chosen arbitrarily small, it follows that $x \in \overline{\sim(-A_1)}$. $\square$

The next lemma uses Specker sequences as it relies on Proposition 3. The proof of the proposition that follows it refers directly to an enumeration of the Turing machines.

Lemma 31. *Let (a_n) be a binary sequence with at most one term equal to 1. If $u < v$, then there exists an open subset I of (u, v), such that*

if $a_n = 1$, then $\rho(u, I) < (v - u)/n$;

if $\rho(u, I) > 0$, then either $a_n = 1$ for some n, or $a_n = 0$ for all n; and $\partial(-I)$ is empty.

Proof. We may assume that $u = 0$ and $v = 1$. By Proposition 3, for each n there is an inhabited open subset J_n of $(1/(n+1), 1/n)$ with $\partial(-J_n)$ empty. Let

$$I = \bigcup \{J_n : a_n = 1\}.$$

Clearly, I is inhabited if and only if $a_n = 1$ for some n; in which case, $I = J_n$ and $\rho(0, I) < 1/n$. If $\rho(0, I) > 0$, then choose N such that $\rho(0, I) > 1/N$; either $a_n = 0$ for each $n \leq N$, in which case $a_n = 0$ for all n, or else $a_n = 1$ for some $n \leq N$.

If $x \in \partial(-I)$, and $a_n = 1$ for some n, then $I = J_n$, a contradiction. So $a_n = 0$

for all n, and therefore I is empty; this is impossible as $x \in \partial(-I)$. So $\partial(-I)$ is empty. $\square$

Proposition 32 [recursive counterexample]. *There exists $\Omega \subset \boldsymbol{R}$ that is open and coherent, has finite boundary, and is not edge coherent.*

Proof. Let $a_{mn} = 1$ if the m^{th} Turing machine halts on the input m at step n, and $a_{mn} = 0$ otherwise. Construct

$$I_m \subset \left(\frac{1}{m+1}, \frac{1}{m} - \frac{1}{(m+1)^2}\right)$$

from the sequence $a_{m1}, a_{m2}, a_{m3}, \ldots$ as in Lemma 31, and set

$$\Omega \equiv (0,1) - \bigcup_m I_m = -\left((-\infty, 0] \cup [1, \infty) \cup \bigcup_m I_m\right).$$

Then Ω is coherent and open, and $\{0,1\} \subset \partial\Omega$. Given $x \in (0,1) \cap \partial\Omega$, choose m such that $x \in (1/(m+2), 1/m)$, and set

$$L \equiv \left(-\infty, \frac{1}{m+2} - \frac{1}{(m+3)^2}\right] \cup \left[\frac{1}{m} + \frac{1}{m^2}, \infty\right),$$

$$R \equiv (-\infty, 0] \cup [1, \infty) \cup \bigcup_{n \neq m, m+1} I_n.$$

Then $\Omega = -(I_m \cup I_{m+1} \cup R)$, $R \subset L$, and $I_m \cup I_{m+1} \subset (1/(m+2), 1/m)$. Noting that L is closed and located, and that

$$\rho\left(L, \left(\frac{1}{m+2}, \frac{1}{m}\right)\right) > 0,$$

we now apply Proposition 30 to show that

$$\partial\Omega = \partial(-(I_m \cup I_{m+1})) \cup \partial(-R).$$

But $\partial(-R) \subset L$, by Proposition 29, and $x \notin L$, so (again by Proposition 30)

$$x \in \partial(-(I_{m+1} \cup I_m)) = \partial(-I_{m+1}) \cup \partial(-I_m) = \emptyset.$$

Hence $\partial\Omega = \{0,1\}$.

If $1/(m+1) \in \Omega$, then $\rho(1/(m+1), I_m) > 0$, because $\Omega \subset -I_m$. Then, by Lemma 31, we can decide the halting behaviour of the m^{th} Turing machine on the input m. It follows that if Ω were edge coherent, then, since $1/(m+1)$ is bounded away from $\partial\Omega$ and belongs to $\bar{\Omega}$ for each m, we could solve the restricted halting problem. $\square$

Let Ω be a bounded metric complement of a located set. We are interested in approximating Ω by a compact set $K \subset\subset \Omega$. Proposition 19 shows that we can always approximate Ω in terms of the metric. Our final example shows, however, that even when Ω is integrable, we need not be able to approximate Ω in measure.

To show this, we construct a particular sequence which is eventually bounded away from any given real number. Unlike a Specker sequence, this one enumerates a compact set and is not monotone. Our construction starts with a singular cover of $\boldsymbol{R}$, the existence of which is a consequence of Church's thesis.

Proposition 33 [recursive counterexample]. *For each $\varepsilon > 0$ there is a countable compact set L of rational numbers in $[0, 1]$ such that every point of L is isolated, and $B(L, \delta)$ has measure greater than $1 - \varepsilon$ for each $\delta > 0$.*

Proof. Let (I_n) be a sequence of open intervals with rational endpoints that covers $\boldsymbol{R}$ and satisfies $\sum_{n=1}^{m} |I_n| < \varepsilon$ for each m. Define

$$L_n \equiv \left\{ \frac{i}{2^n} \in [0, 1] : \left(\frac{i}{2^n}, \frac{i+1}{2^n} \right) \text{ is not contained in } I_1 \cup \cdots \cup I_n \right\},$$

and let L be the union of the L_n. Because each I_k has rational endpoints, the set L_n is finite. If $i/2^n \in L_m$ for $m > n$, then $i/2^n \in L_n$, so L is a detachable subset of $\boldsymbol{Q}$ – that is, for each $q \in \boldsymbol{Q}$ either $q \in L$ or $q \notin L$.

To show that L is totally bounded, we show that the set $L_0 \cup \cdots \cup L_n$ is a 2^{-n}-approximation to L. Suppose that $i/2^m \in L_m$ and $m > n$. There exists j such that

$$(*) \qquad \left(\frac{i}{2^m}, \frac{i+1}{2^m} \right) \subset \left(\frac{j}{2^n}, \frac{j+1}{2^n} \right)$$

so $j/2^n$ is in L_n and is within 2^{-n} of $i/2^m$.

If r is any real number, then $r \in I_k$ for some k. As I_k is open, we can choose $n \geq k$ such that $B(r, 2^{-n}) \subset I_k$; whence $B(r, 2^{-n}) \subset I_1 \cup \cdots \cup I_n$. We will show, by induction on m, that

$$(**) \qquad L_m \cap B(r, 2^{-n-1}) \subset L_0 \cup \cdots \cup L_n.$$

Let $x = i/2^m$ belong to $L_m \cap B(r, 2^{-n-1})$ with $m > n$. If i is even, then $(**)$ holds by induction. We shall show that i cannot be odd. To this end, assume that i is odd and choose j as in $(*)$. Then

$$\rho\left(\frac{i}{2^m}, \left\{ \frac{j}{2^n}, \frac{j+1}{2^n} \right\} \right) \geq 2^{-n-1} > \left| \frac{i}{2^m} - r \right|,$$

so r must be in $(j/2^n, (j+1)/2^n)$. Hence

$$\begin{aligned} \left(\frac{i}{2^m}, \frac{i+1}{2^m} \right) &\subset \left(\frac{j}{2^n}, \frac{j+1}{2^n} \right) \\ &\subset B(r, 2^{-n}) \\ &\subset I_1 \cup \cdots \cup I_n \\ &\subset I_1 \cup \cdots \cup I_m, \end{aligned}$$

contradicting the fact that $x \in L_m$. This completes the inductive proof of $(**)$. It follows that for each real number r there exists n such that

$$L \cap B(r, 2^{-n-1}) \subset L_0 \cup \cdots \cup L_n;$$

whence r is bounded away from all but finitely many elements of L. This shows that L is closed and therefore compact, and that each element of L is isolated.

Given $\delta > 0$, choose n so that $2^{-n} < \delta$. If x is an irrational point of $[0, 1] - (I_1 \cup \cdots \cup I_n)$, then there exists a unique i such that $x \in (i/2^n, (i+1)/2^n)$; so

$$x \in \left(\frac{i}{2^n}, \frac{i+1}{2^n}\right) - (I_1 \cup \cdots \cup I_n)$$

and therefore $i/2^n \in L_n$. Since $|x - i/2^n| < \delta$, it follows that $B(L, \delta)$ contains each irrational point of $[0, 1] - (I_1 \cup \cdots \cup I_n)$. Hence $B(L, \delta)$ has measure greater than $1 - \varepsilon$. $\square$

Although the set $\Omega \equiv (0, 1) - L$, being integrable with positive measure, can be approximated in measure by compact sets that are contained in it ([2], Chapter 6, (6.7)), the above proposition shows that it cannot be approximated in measure by compact sets that are *well* contained in it. Contrast this to the classical situation in which you can prove that any compact subset of an open set is well contained in it.

The set L, even without its measure-theoretic properties, can be used to construct other standard pathological recursive examples. We get an ascending Specker sequence by setting

$$r_{n+1} \equiv \min\{L \backslash \{r_1, \ldots, r_n\}\}.$$

To construct a positive Lipschitz function f on $[0, 1]$ with infimum 0, let (ℓ_n) enumerate L, and for each n choose $\delta_n > 0$ such that $B(\ell_n, 3\delta_n) \cap L = \{\ell_n\}$. Define

$$f_n(x) \equiv \sup\{0, \delta_n - |x - \ell_n|\}$$

and

$$f(x) \equiv \rho(x, L) \vee \sup_{n \geq 1} f_n(x).$$

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